Eur. Phys. J. B **29**, 605–611 (2002)
DOI: 10.1140/epjb/e2002-00343-2

THE EUROPEAN PHYSICAL JOURNAL B

Separable-entangled frontier in a bipartite harmonic system

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Received 18 April 2002 / Received in final form 11 July 2002
Published online 31 October 2002 –

Abstract. We consider a statistical mixture based on that of two identical harmonic oscillators which is characterized by four parameters, namely, the concentrations (x and y) of diagonal and nondiagonal bipartite states, and their associated thermal-like noises (T/α and T, respectively). The fully random mixture of two spins 1/2 as well as the Einstein-Podolsky-Rosen (EPR) state are recovered as particular instances. By using the conditional nonextensive entropy as introduced by Abe and Rajagopal, we calculate a bound for the separable-entangled frontier. Although this procedure is known to provide a necessary but in general *not* sufficient condition for separability, it does recover, in the particular case $x = T = 0$ ($\forall\alpha$), the 1/3 exact result known as Peres' criterion. The $x = 0$ frontier remarkably resembles to the critical line associated with standard diluted ferromagnetism where the entangled region corresponds to the ordered one and the separable region to the paramagnetic one. The entangled region generically shrinks for increasing T or increasing α.

PACS. 03.65.Bz Foundations, theory of measurement, miscellaneous theories (including Aharonov-Bohm effect, Bell inequalities, Berry's phase) – 03.67.-a Quantum information – 05.20.-y Classical statistical mechanics – 05.30.-d Quantum statistical mechanics

Quantum entanglement is a quite amazing physical phenomenon, and has attracted intensive interest in recent years due to its possible applications in quantum computation, teleportation and cryptography, as well as to its connections to quantum chaos [1–19]. A nonextensive statistical mechanics [20] was proposed in 1988 by one of us, and is currently applied [21–26] to a variety of thermodynamically anomalous systems which, in one way or another, exhibit (multi) fractals aspects. Among these anomalous systems, a prominent place is occupied by systems including long-range interactions and Lévy distributions. So being, it is after all not surprising that this thermostatistical formalism has interesting implications [27–29] in the area of quantum entanglement and its intrinsic nonlocality, thus showing the confluence of two concepts coming from distinct physical areas.

Quantum systems can be more or less entangled, which makes relevant the discussion of whether a given system is or not separable. Separability, which we shall define in detail later on, is a crucial feature in the discussion on whether a quantum physical system is susceptible of a *local realistic* description with hidden variables. These issues were first discussed in 1935 by Einstein, Podolsky and Rosen (EPR) [30] and by Schrödinger [31], and since then by many others [1–19]. As we have mentioned above we shall see that entropic nonextensivity provides a path through which it is possible to discuss quantum entanglement [27,29,32].

In this work we want to analyze the effect that external (thermal) noise may produce on a quantum device, *i.e.*, the influence of thermal-like noise over the entanglement of a given state. For this goal we study a composite system of two harmonic oscillators with identical energy spectra. The mixed state we shall consider for this bipartite system involves Boltzmann-like probabilities (through the temperature parameter T) as well as a few additional terms in such a way that the bipartite spin 1/2 system is recovered as a particular case. More precisely, the 1/3 Peres' criterion for separability will emerge as the $T = 0$ limit of this specific mixed state.

Following along the lines of Abe and Rajagopal [27], we use the nonextensive entropy $S_q \equiv \frac{1-\mathrm{Tr}\rho^q}{q-1}$ ($q \in \mathcal{R}$; $S_1 = -\mathrm{Tr}\,\rho\ln\rho$), ρ being the density matrix, in order to study the frontier between *separable* (*quantum nonentangled*) and *nonseparable* (*quantum entangled*) regions. More precisely, we determine a frontier in some parameter-space which is either the exact one or an *overestimation* of the separable region. We remind that the entropic arguments that are used within this approach provide, like the Peres' partial transpose method [12], necessary but *not* sufficient conditions for separability.

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In order to make this work self-contained we will present, in the following, the connection between the nonextensive statistical mechanics and quantum entanglement. Let us begin with the definition of a generalized entropy for a quantum system

$$S_q = \frac{1 - \mathrm{Tr}\rho^q}{q-1} \quad (q \in \Re;\ \mathrm{Tr}\rho = 1;\ S_1 = -\mathrm{Tr}\ \rho \ln \rho), \quad (1)$$

where ρ is the density operator. Let us assume now that ρ is the density operator associated with a composed system $A+B$. Then the marginal density operators are given by $\rho_A \equiv \mathrm{Tr}_B\ \rho$ and $\rho_B \equiv \mathrm{Tr}_A\ \rho$ ($\mathrm{Tr}_A\ \rho_A = \mathrm{Tr}_B\ \rho_B = 1$). The systems A and B are said to be *uncorrelated* (or *independent*) if and only if

$$\rho = \rho_A \otimes \rho_B. \quad (2)$$

Otherwise they are said to be *correlated*. Two correlated systems are said to be *separable* (or *unentangled*) if and only if it is possible to write ρ as follows:

$$\rho = \sum_{i=1}^{W} p_i\ \rho_A^{(i)} \otimes \rho_B^{(i)} \qquad \left(p_i \geq 0\ \forall i;\ \sum_{i=1}^{W} p_i = 1\right) \cdot \quad (3)$$

The limiting case of independency is recovered for *certainty*, *i.e.*, if all p_i vanish excepting one which equals unity. *Nonseparability* (or *entanglement*) is at the basis of the amazing phenomena mentioned before and, as already pointed, at the center of the admissibility of a *local realistic* description of the system in terms of hidden variables. As it is known correlation is a concept which is meaningful both classically and quantically. Entanglement, in the present sense, is meaningful only within quantum mechanics.

The characterization of quantum entanglement is not necessarily simple to implement, since it might be relatively easy in a specific case to exhibit the form of equation (3), but it can be nontrivial to prove that it *cannot* be presented in that form. Consequently, along the years appreciable effort has been dedicated to the establishment of general operational criteria, preferentially in the form of necessary and sufficient conditions whenever possible. Peres [12] pointed out a few years ago a *necessary* condition for separability, namely the nonnegativity of the partial transpose of the density matrix. In some simple situations (like the simple mixed state of two spin 1/2) Peres' criterion is now known to also be a *sufficient* condition. But, as soon as the case is slightly more complex (*e.g.*, 3×3 or 2×4 matrices) it is known now to be *insufficient*.

Within this scenario Abe and Rajagopal [27] recently proposed a different condition, claimed to be a *necessary* one, based on the nonextensive entropic form S_q given in equation (1). Let us summarize the idea. The quantum version of conditional probabilities is not easy to formulate in spite of being so simple within a classical framework. The difficulties come from the fact that generically ρ does not commute with either ρ_A or ρ_B. Consistently, the concept of *quantum conditional entropy* is a sloppy one. Abe and Rajagopal suggested a manner of shortcutting this difficulty, namely through the adoption, of the following definitions of the conditional entropies $S_q(A|B)$ and $S_q(B|A)$, respectively given by

$$S_q(A|B) \equiv \frac{S_q(A+B) - S_q(B)}{1 + (1-q)S_q(B)}, \quad (4)$$

and

$$S_q(B|A) \equiv \frac{S_q(A+B) - S_q(A)}{1 + (1-q)S_q(A)}, \quad (5)$$

where $S_q(A+B) \equiv S_q(\rho)$, $S_q(A) \equiv S_q(\rho_A)$ and $S_q(B) \equiv S_q(\rho_B)$. Obviously, for the case of independence, *i.e.*, when $\rho = \rho_A \otimes \rho_B$, these expressions lead to $S_q(A|B) = S_q(A)$ and $S_q(B|A) = S_q(B)$, known to be true also in quantum mechanics.

Both classical and quantum entropies $S_q(A+B)$, $S_q(A)$ and $S_q(B)$ are always nonnegative. This is not the case of the conditional entropies $S_q(A|B)$ and $S_q(B|A)$, which are always nonnegative classically, but which can be negative quantically (see also [33]). It is therefore natural to expect that separability implies nonnegativity of the conditional entropies for all q. This is the criterion proposed in [27].

Peres' criterion seems to be less restrictive than the entropic one in the sense that it might provide a larger quantum entangled region. Although we have no proof that it cannot be the other way around, we have not encountered such an example. In several cases, including some of increasingly many spins, both criteria have produced the same result, presumably the asymptotically exact answer. Details can be found in [2,11–13,19,34]. As a general trend, it seems that the transposed matrix criterion *overestimates the separable region* (*i.e.*, *underestimates the quantum entangled region*) less or equally than the conditional entropy criterion does. It could well be that whenever both criteria produce the same result, this result is the exact answer [35].

Equations (4, 5) have in fact been proved for a classical system [32], not for a quantum one. It seems however plausible [27] that they preserve the same form in both classical and quantum cases. Such conditional entropies enable what is a necessary condition for separability. Imposing the conditions $S_q(A|B), S_q(B|A) > 0$ for all values of q, the particular space of parameters becomes divided into two regions: one where any conditional entropy S_q is positive and the other one – a domain of entangled states – where at least one conditional entropy is negative for some q. Therefore, a line (or a frontier, generally speaking) like a critical one, emerges. By the way, an interesting related point is that of defining a measure of the degree of entanglement. As already advanced in [29,32] one can define an entanglement "order parameter" which plays a role analogous to the order parameter in standard critical phenomena (see also [36]).

To establish the separable-nonseparable frontier we are looking for, it will become clear that the Boltzmann-Gibbs-Shannon entropy ($q = 1$) is a concept too poor for properly discussing quantum entanglement, a conclusion recently reached also by Brukner and Zeilinger [28] from a different path. An appreciable amount of arguments are now available in the literature which connect quantum entanglement and thermodynamics [5,10,11,15–17,28,36].

Here we address a system of two independent harmonic oscillators A and B to study the effect of thermal noise, as we will introduce soon. Additionally, this system allows to study the effect of an unbounded spectrum which is known to produce significant differences at the thermodynamic level. For the oscillators we consider the basis $|n\rangle_A|m\rangle_B \equiv |n,m\rangle$, where $n,m = 0,1,2,...$ Let us define the symmetric and antisymmetric states $|n,m^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|n,m\rangle \pm |m,n\rangle)$ for $n > m$. These states for the two oscillators system are clearly isomorphic to the two spins EPR state. We construct first a density matrix of the form

$$\rho_{A+B} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} C_{nm}|n,m^-\rangle\langle n,m^-| \,, \tag{6}$$

where we choose $C_{nm} \propto \mathrm{e}^{-(n+m)/T}$ in order to introduce a temperature T (noise). Since we want to study the effect of noise on a known state, it is interesting to recover in some limit a state isomorphic to the well known Werner-Popescu state [1,7]. So, we generalize the weights of the lower energy eigenstates, a kind of filtering out higher energy states such as encountered in laser physics, obtaining

$$\begin{aligned}\rho_{A+B} = {} & \\ & \frac{1-y}{4}\Big[|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1| + |1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1|\Big] \\ & + y\, b \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \mathrm{e}^{-(n+m)/T}|n,m^-\rangle\langle n,m^-|,\end{aligned} \tag{7}$$

with

$$b \equiv 1 \Bigg/ \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \mathrm{e}^{-(n+m)/T} = 2\left(1 - \mathrm{e}^{-1/T}\right)\sinh(1/T)\,\cdot \tag{8}$$

Finally, we include diagonal states ($|n,n\rangle$) with thermal weights (not necessarily equal to those of the antisymmetric component) as this further generalization does not increase the complexity of the computations and can give additional information. Thus, we have:

$$\begin{aligned}\rho_{A+B} = {} & \frac{1-x-y}{4} \\ & \times \Big[|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1| + |1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1|\Big] \\ & + x\, a \sum_{n=2}^{\infty} \mathrm{e}^{-2n\alpha/T}|n,n\rangle\langle n,n| + y\, b \\ & \times \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \mathrm{e}^{-(n+m)/T}|n,m^-\rangle\langle n,m^-|,\end{aligned} \tag{9}$$

where $T \geq 0$, $\alpha \geq 0$ and

$$a \equiv 1 \Bigg/ \sum_{n=2}^{\infty} \mathrm{e}^{-2n\alpha/T} = (1 - \mathrm{e}^{-2\alpha/T})\,\mathrm{e}^{4\alpha/T}\,. \tag{10}$$

Parameter α measures the importance of the noise associated with the nondiagonal terms with respect to the noise associated with the diagonal ones. An increase of the temperature-like measure of noise T incorporates higher energy levels in the mixed state under consideration.

We easily verify that $\mathrm{Tr}\rho_{A+B} = 1$. Since the eigenvalues of ρ_{A+B} must be numbers within $[0,1]$, we have $0 \leq x, y \leq 1$ and $x + y \leq 1$.

Let us emphasize that the mixed state given by equation (9) is not generically a thermal equilibrium state, although it can be close to equilibrium for appropriate values of the parameters (x,y). Notice also that, for $x = y = 0$, we have

$$\begin{aligned}\rho_{A+B} = {} & \\ & \frac{1}{4}\Big[|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1| + |1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1|\Big] \\ & = \frac{1}{4}\hat{1}_4,\end{aligned} \tag{11}$$

where $\hat{1}_4$ is the 4-dimensional identity matrix. In other words, this state is isomorphic to a fully random state of two spins 1/2. Notice finally that, for $T = 0$ and $x = 0$, we have an state isomorphic to the Werner-Popescu one [1,7] and if additionally we set $y = 1$ we have $\rho_{A+B} = |1,0^-\rangle\langle 1,0^-|$, which is isomorphic to the celebrated EPR state. The approximate separable-entangled frontier for the $T = x = 0$ particular case is known [12,27,29,32], namely $y = 1/3$ ($\forall\alpha$). Before we address the discussion of the full frontier in the (x,y,T,α)-space, it is convenient to rewrite equation (9) in the following equivalent form:

$$\begin{aligned}\rho_{A+B} = {} & \frac{1-x-y}{4} \\ & \times \Big[|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1| + |1,0^+\rangle\langle 1,0^+|\Big] \\ & + x\, a \sum_{n=2}^{\infty} \mathrm{e}^{-2n\alpha/T}|n,n\rangle\langle n,n| \\ & + \left(\frac{1-x-y}{4} + y\, b\, \mathrm{e}^{-1/T}\right)|1,0^-\rangle\langle 1,0^-| \\ & + y\, b \sum_{n=2}^{\infty}\sum_{m=0}^{n-1} \mathrm{e}^{-(n+m)/T}|n,m^-\rangle\langle n,m^-|\,\cdot\end{aligned} \tag{12}$$

Due to the orthonormality of all the states appearing in this expression, the calculation of $(\rho_{A+B})^q$ is easily carried out. It follows

$$
\begin{aligned}
\mathrm{Tr}(\rho_{A+B})^q &= 3\left(\frac{1-x-y}{4}\right)^q + \frac{x^q\left(1-\mathrm{e}^{-2\alpha/T}\right)^q}{1-\mathrm{e}^{-2q\alpha/T}} \\
&+ \left(\frac{1-x-y}{4} + 2y\left(1-\mathrm{e}^{-1/T}\right)\mathrm{e}^{-1/T}\sinh(1/T)\right)^q \\
&+ \frac{\left(2y\,(1-\mathrm{e}^{-1/T})\mathrm{e}^{-1/T}\sinh(1/T)\right)^q\left(1+\mathrm{e}^{-q/T}-\mathrm{e}^{-2q/T}\right)}{2\sinh(q/T)\left(1-\mathrm{e}^{-q/T}\right)}\cdot
\end{aligned}
\tag{13}
$$

Let us now address the calculation of $(\rho_B)^q$. We have

$$
\begin{aligned}
\rho_B \equiv \mathrm{Tr}_A\rho_{A+B} &= \left(\frac{1-x-y}{2} + y\mathrm{e}^{-1/T}\sinh(1/T)\right)|0\rangle\langle 0| \\
&+\left(\frac{1-x-y}{2} + y\sinh(1/T)\left(\mathrm{e}^{-1/T}-\mathrm{e}^{-2/T}+\mathrm{e}^{-3/T}\right)\right)|1\rangle\langle 1| \\
&+\sum_{n=2}^{\infty}\left(x\left(1-\mathrm{e}^{-2\alpha/T}\right)\mathrm{e}^{-2(n-2)\alpha/T}\right. \\
&\left.+y\sinh(1/T)\left(\mathrm{e}^{-n/T}-\mathrm{e}^{-2n/T}+\mathrm{e}^{-(2n+1)/T}\right)\right)|n\rangle\langle n|,
\end{aligned}
\tag{14}
$$

where $|n\rangle$ denotes the nth state of oscillator B. It now follows

$$
\begin{aligned}
\mathrm{Tr}(\rho_B)^q &= \left(\frac{1-x-y}{2} + y\mathrm{e}^{-1/T}\sinh(1/T)\right)^q \\
&+\left(\frac{1-x-y}{2} + y\sinh(1/T)\left(\mathrm{e}^{-1/T}-\mathrm{e}^{-2/T}+\mathrm{e}^{-3/T}\right)\right)^q \\
&+\sum_{n=2}^{\infty}\left(x\left(1-\mathrm{e}^{-2\alpha/T}\right)\mathrm{e}^{-2(n-2)\alpha/T}\right. \\
&\left.+y\sinh(1/T)\left(\mathrm{e}^{-n/T}-\mathrm{e}^{-2n/T}+\mathrm{e}^{-(2n+1)/T}\right)\right)^q\cdot
\end{aligned}
\tag{15}
$$

Now that we have $\mathrm{Tr}(\rho_{A+B})^q$ and $\mathrm{Tr}(\rho_B)^q$ as explicit functions of (x, y, T, α), we can immediately obtain $S_q(A|B)(= S_q(B|A))$ through the use of equations (1, 4) yielding

$$
S_q(A|B) = \frac{1}{q-1}\left(1 - \frac{\mathrm{Tr}(\rho_{A+B})^q}{\mathrm{Tr}(\rho_B)^q}\right)\cdot \tag{16}
$$

$S_q(A|B)$ as a function of (x, y) for typical values of (q, T) and $\alpha = 1$ is represented in Figures 1 and 2. By comparing these figures, notice the nonuniform convergence at $T = 1/q = 0$.

Figure 3 exhibits, in (y, q) space, the line verifying $S_q(A|B) = 0$, for particular values of (x, T, α). The physical region is the one satisfying $0 \le x + y \le 1$. Given x, for y above this critical line and within the physical region $y \le 1 - x$, one has $S_q(A|B) < 0$, which signals the existence of an entangled state. Notice that for sufficiently small x, the strongest restriction to the separable region is imposed at $q \to \infty$. As x increases, the minimum of y occurs at a finite q, q_{min} (the possible nonmonotonicity of $S_q(A|B)$ with regard to q has already been noticed by [37]). Since the strongest restriction to separability corresponds to the nonnegativity of both $S_{q_{min}}(A|B)$ and $S_{q_{min}}(B|A)$, then this condition is the one which better

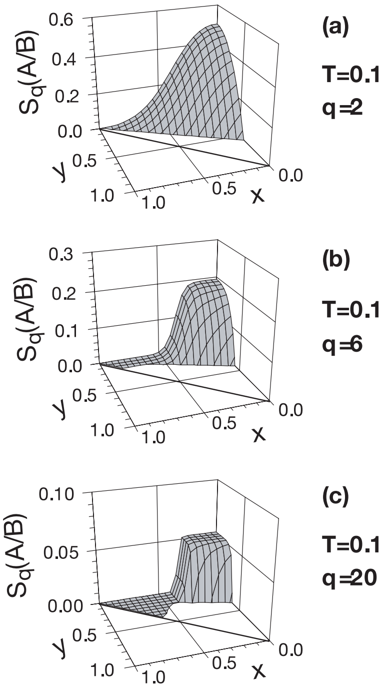

Fig. 1. $S_q(A|B)$ as a function of (x, y) for $\alpha = 1$, $T = 0.1$ and typical values of q.

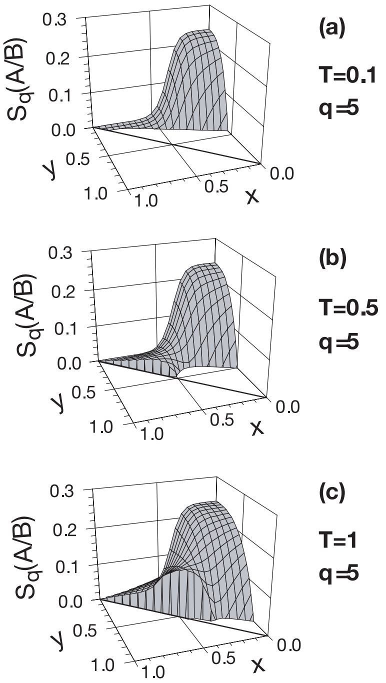

Fig. 2. $S_q(A|B)$ as a function of (x, y) for $\alpha = 1$, typical values of T and $q = 5$.

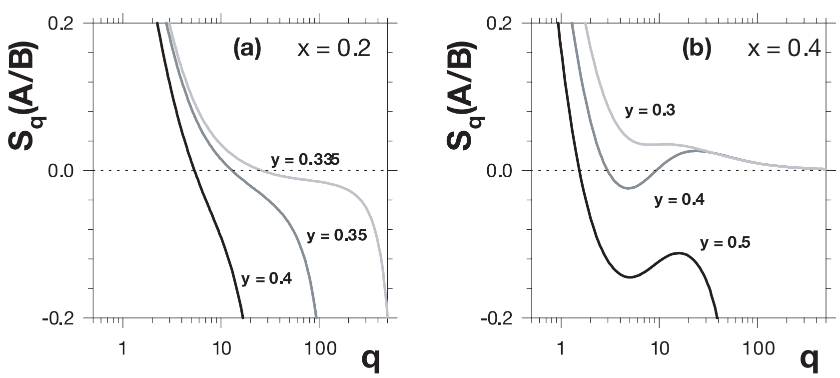

Fig. 3. The line $S_q(A|B) = 0$, in (y, q)-space, for particular values of x, $T = 0.5$ and $\alpha = 1$.

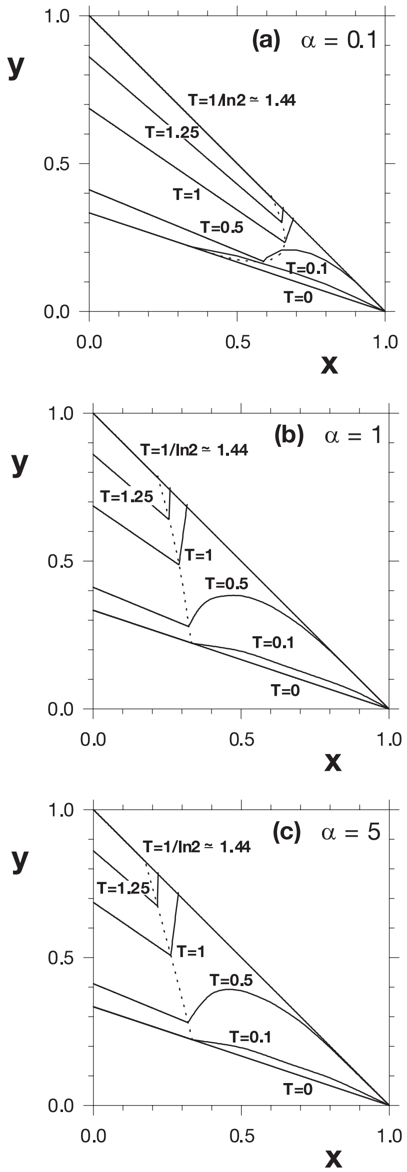

Fig. 4. Frontier of the region where $S_q(A|B) < 0$ for some q, in the (x, y)-space for typical values of T and $\alpha = 0.1$ (a), 1 (b), 5 (c). The dotted lines correspond to the locus of the vertices of the frontier for any T. Notice that the condition $S_q(A|B) < 0$ for some value of q guarantees entanglement.

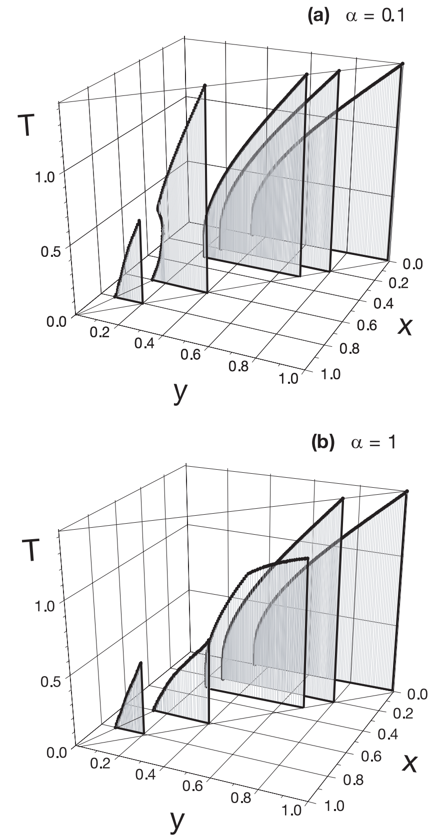

Fig. 5. Frontier of the region where $S_q(A|B) < 0$ for some q, in the (x, y, T)-space for fixed values of x and α= 0.1 (a), 1 (b). The entire plane $x + y = 1$ ($\forall T$) belongs to the separable-entangled frontier.

approaches the separable-entangled frontier we are looking for and which is represented in Figure 4. The entangled region in the (x, y, T)-space is illustrated in Figure 5 for various values of (x, α). For $T = 0$ and arbitrary α, the system is entangled if $(1-x)/3 < y \leq 1-x$, which immediately recovers the Peres' 1/3 criterion for $x = 0$. As T increases, the entangled domain shrinks against the border $x + y = 1$ and disappears at a critical T, T_c ($T_c = 1/\ln 2$ for $x = 0$).

If the conditional entropy $S_q(A|B)$ is a monotonically decreasing function of q, then the strongest case corresponds to the nonnegativity of $S_\infty(A|B)$. Whenever the frontier is fully defined by $q \to \infty$, the system is entangled in the region defined by

$$y^* < y \leq 1 - x \quad \text{and} \quad x < x^* \tag{17}$$

where

$$y^* \equiv \frac{1-x}{3 - 2\,\mathrm{e}^{-1/T}(2 + \mathrm{e}^{-1/T} - 2\,\mathrm{e}^{-2/T})} \tag{18}$$

and

$$x^* \equiv \frac{\left(2(1-\mathrm{e}^{-2/T})(2 - 3\,\mathrm{e}^{-1/T} + \mathrm{e}^{-3/T} - \mathrm{e}^{-4/T}) - 1\right) y + 1}{(5 - 4\,\mathrm{e}^{-2\alpha/T})}\,. \tag{19}$$

However, for sufficiently small T, there is a curved line (corresponding to the nonlinear portion of the frontier in Fig. 4), given by $y = f(x)$, which is defined from the conditional entropy for finite q_{min} and that can be numerically determined. In this case, the entangled region is given by

$$y^* < y \leq 1 - x \quad \text{and} \quad y > f(x). \tag{20}$$

An increase of the noise parameter T (see Figs. 4 and 5), *i.e.*, an increase of the relative weight of the high energy states of the oscillators, causes shrinking of the entangled region. This is intuitively expected. Also, interestingly enough, decreasing the parameter α (see Figs. 4 and 5), which corresponds to an increase of the noise associated with the diagonal states, causes the entangled region to enlarge. This is not surprising after all, since this noise makes the diagonal terms to become less effective than the off-diagonal ones, which makes quantum coherence to be relatively stronger. The interplay of these effects may cause an interesting re-entrance, as illustrated in Figure 5a for the value $x = 0.6$. In this situation, corresponding in fact to small values of α, it is possible to loose the quantum entanglement as T increases, and then recover it back, until definitive loss for very large T. To conclude, we believe that the present results provide quantitative insight on the influence of the external world (here represented by two different types of noise) on a quantum device. Our results could guide the engineering of entangled states in the laboratory.

One of us (DP) acknowledges warm hospitality at the Centro Brasileiro de Pesquisas Físicas. Also, we acknowledge partial financial support from PRONEX, CNPq and FAPERJ.

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